Matchings in Random Bipartite Graphs with Applications to Hashing-based Data Structures

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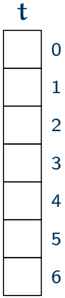
$f(x)$	arbitrary	retrieval DS

Basic Scheme

Structure:

- table t with m cells, each of capacity r bits



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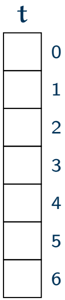

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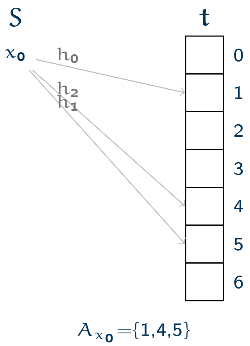

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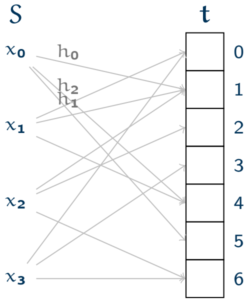

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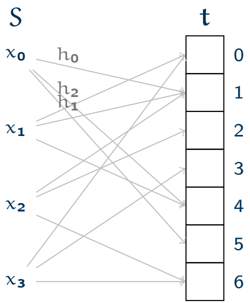

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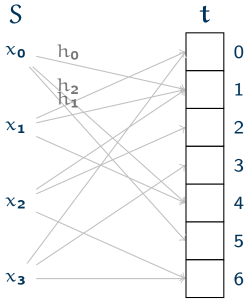

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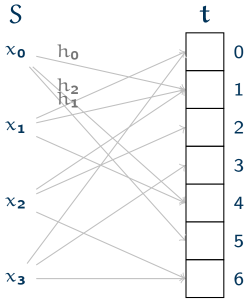

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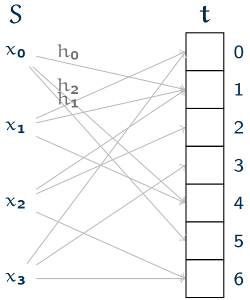

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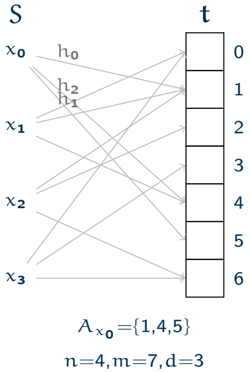
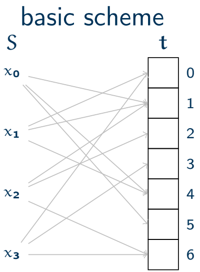
- e.g. [Dietzfelbinger and Rink, 2009] **not a topic here**



Abstract Representations



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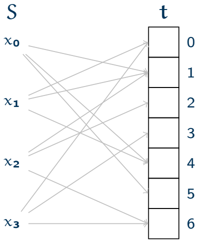
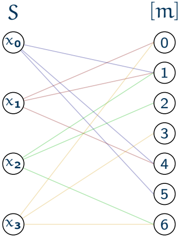


bipartite graph $\mathcal{B}$



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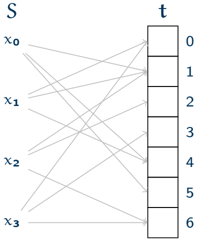

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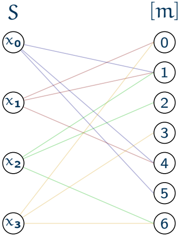
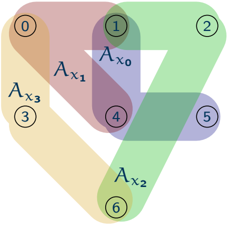


hypergraph $\mathcal{H}$



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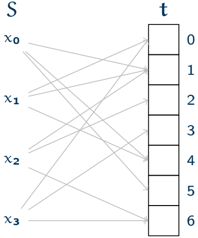

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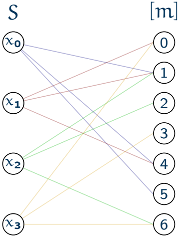

binary matrix $\boldsymbol{M}$

$$
\begin{array}{c}
\\
A_{x_0} \\
A_{x_1} \\
A_{x_2} \\
A_{x_3}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
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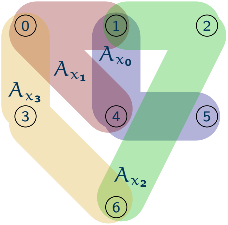
hypergraph $\mathcal{H}$



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- maximum $c = n/m$, $c = c(d)$, such that construction is successful with high probability (fixed d)
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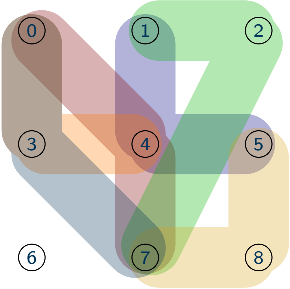
Measurements:

- time for $\texttt{lookup}(\mathfrak{D}, x)$ in number of cell probes
- space complexity in bits
- time complexity in word operations

Outline

Next ...

The 2-Core



Algorithm: Peeling

Input: hypergraph $\mathcal{H} = ([m], E)$

Output: maximum induced sub-hypergraph with minimum degree 2

while $\mathcal{H}$ *has a node* v *of degree* $\leqslant 1$ **do**

  if v *is incident to an edge* A_x **then** delete A_x

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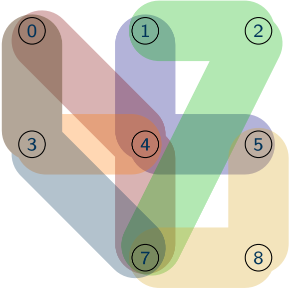

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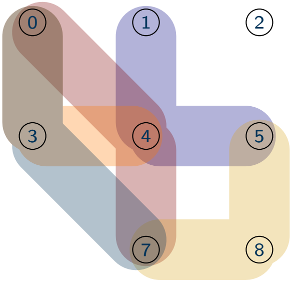

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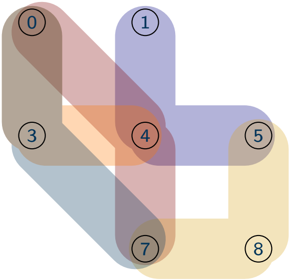

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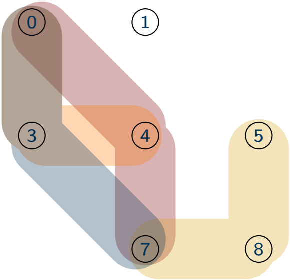

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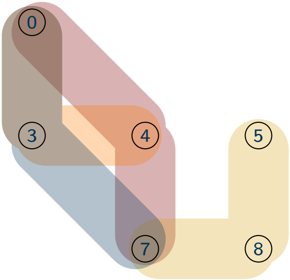

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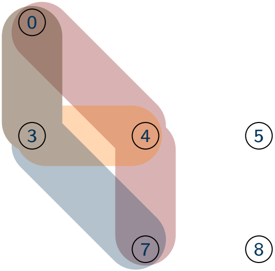

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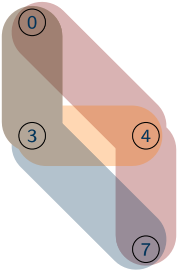

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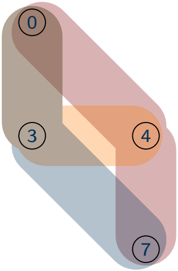

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Analogous procedure in other formulations $\mathcal{B}$ and $\mathbf{M}$ gives the (equivalent) "2-core".

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Theorem: [Molloy, 2004],[Cooper, 2004],[Kim, 2006],...

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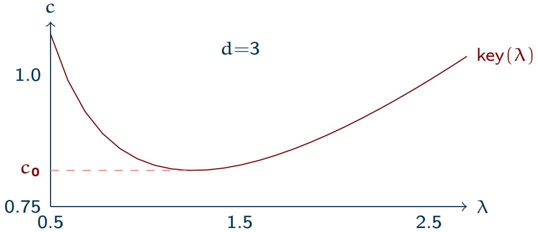

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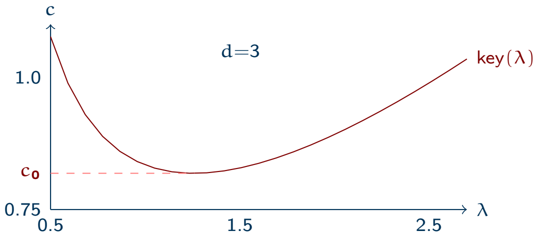

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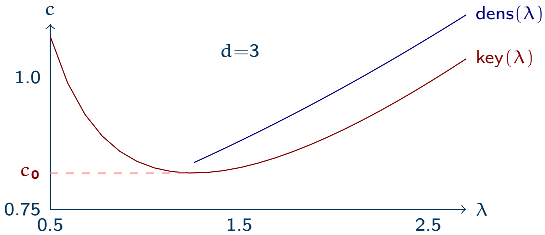

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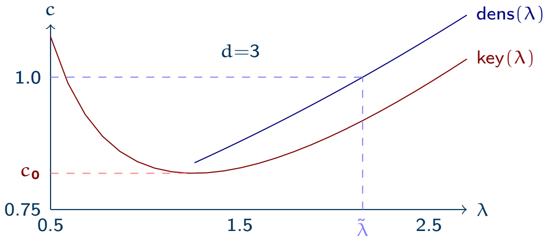

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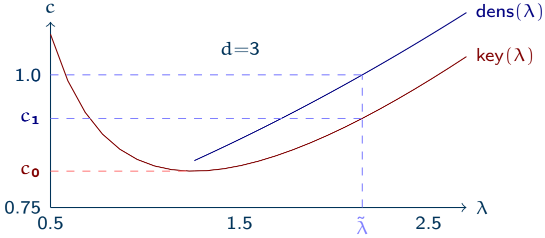

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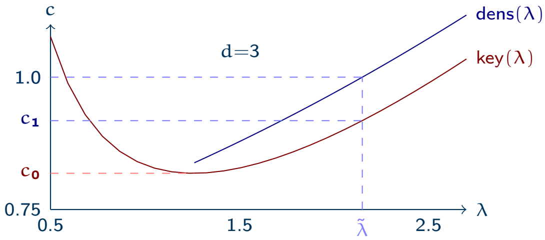


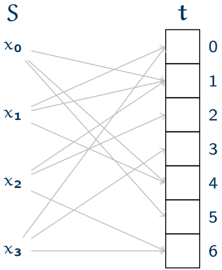
Interested in:
c_0 and c_1, the edge density of $\mathcal{H}$ where the edge density of its 2-core is 1

Next . . .

Multiple Choice Hash Table

d-ary Cuckoo Hashing [Fotakis, Pagh, Sanders, Spirakis, 2003]**:**

- membership tester:



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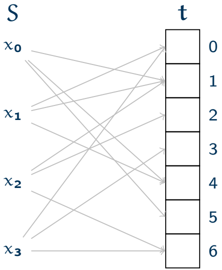

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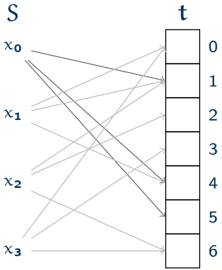

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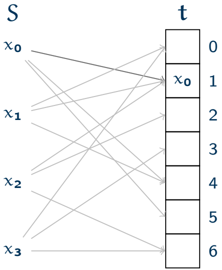

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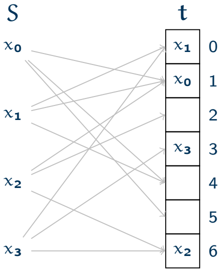

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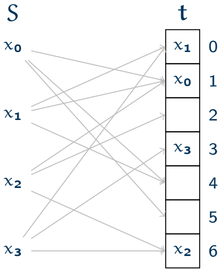

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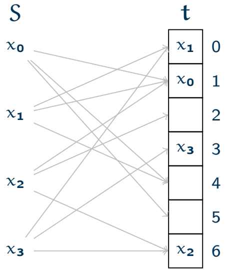

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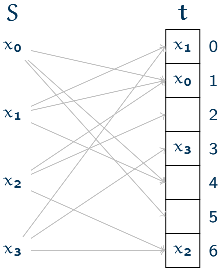

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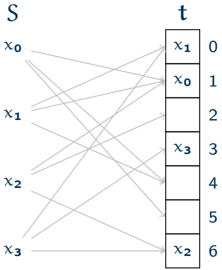

Requirements

Construction possible:

$\stackrel{\text{def}}{=}$ injective mapping $\sigma\colon S \to [m]$,
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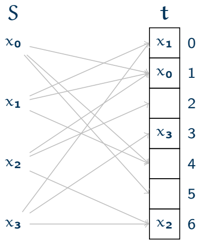
basic scheme



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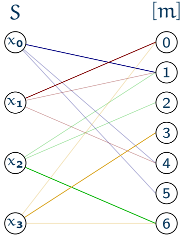

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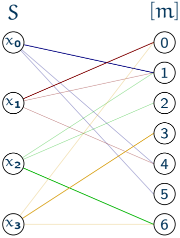

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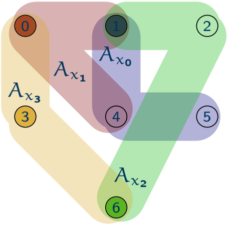

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$\Leftrightarrow$ $n \times n$ submatrix of $\boldsymbol{M} \geqslant$ permutation matrix

binary matrix $\boldsymbol{M}$

$$\begin{array}{c|ccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
A_{x_0} & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
A_{x_1} & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
A_{x_2} & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
A_{x_3} & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
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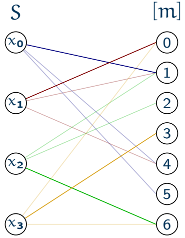

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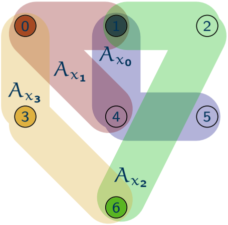

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Theorem: [Fotakis, Pagh, Sanders, Spirakis, 2003]

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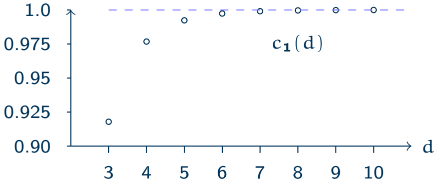

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Question: Given a target average left degree $\overline{d}$, can we improve the success probability using different left degrees d_x compared to using the same left degree?

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Let $\overline{d} > 2$ be the average left degree of $\mathcal{B}$.

- If $\overline{d}$ is integral, then the optimal choice is $d_x = \overline{d}$ for all $x \in S$.
- If $\overline{d}$ is non-integral, then it is optimal if the fraction of nodes with degree $\begin{cases} \lfloor \overline{d} \rfloor \\ \lceil \overline{d} \rceil \end{cases}$ is tightly concentrated around $\begin{cases} \lceil \overline{d} \rceil - \overline{d} \\ \overline{d} - \lfloor \overline{d} \rfloor \end{cases}$.

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Proof idea:

* Degree of each left node is random variable D_x with separate pmf.
* Fix $\mathcal{B}$, but omit 2 left nodes x, y. Compare probability for a matching under slight changes of pmf_x and pmf_y.

Thresholds for Mixed Degrees

Theorem: [Dietzfelbinger, Goerdt, Mitzenmacher, Montanari, Pagh, Rink, 2010]

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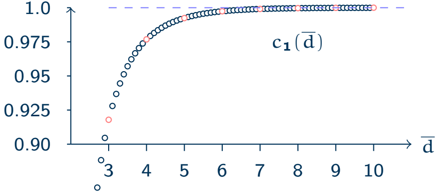
left degrees d_x concentrated around $\overline{d}$



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Question: Can we do better, in linear time?

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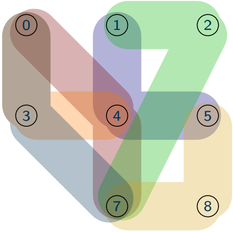
Generalized Selfless Algorithm [Dietzfelbinger et al., 2010]:

- adaption of "Selfless Algorithm" by [Sanders, 2004]



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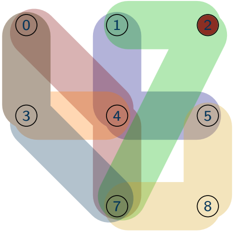

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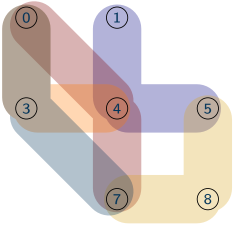

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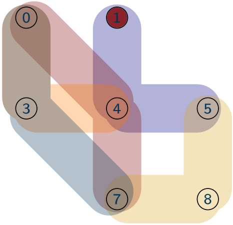

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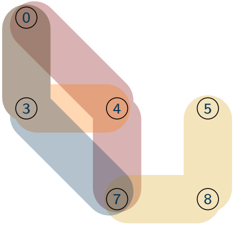

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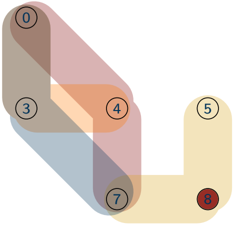

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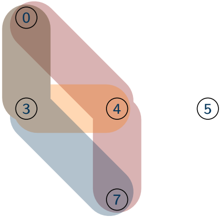

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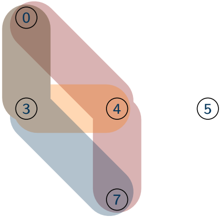

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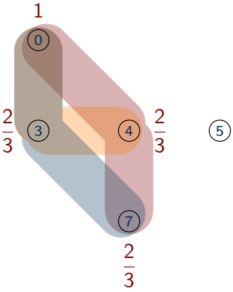

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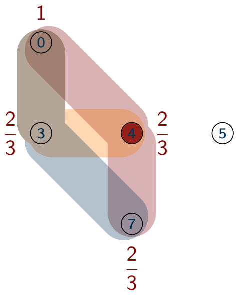

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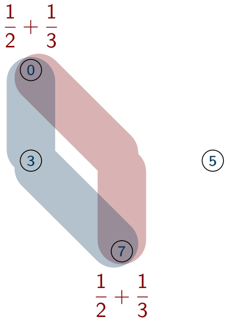

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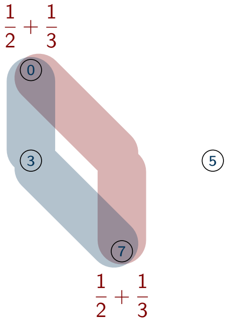

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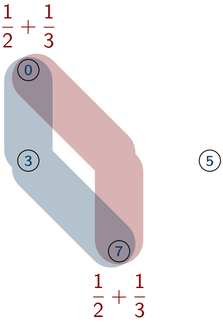

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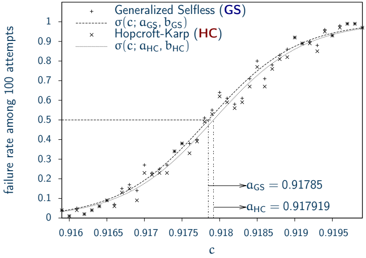
    $\pi(v) = \sum_{A_x \ni v} \frac{1}{|A_x|}$
    if *minimum priority* > 1 **then return** failure
  choose edge $A_x \ni v$ with min cardinality $|A_x|$
  match x and v
  delete A_x and v
until *all edges have been deleted at the end*

Heuristic vs Optimal Algorithm



$m = 10^5$

$d = 3$; theoretical threshold $c_1(d) \approx 0.91794$, interval size 0.004

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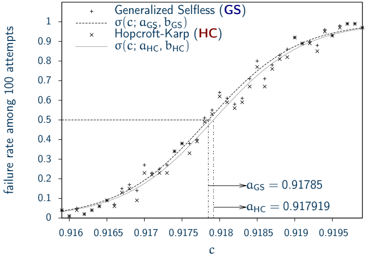

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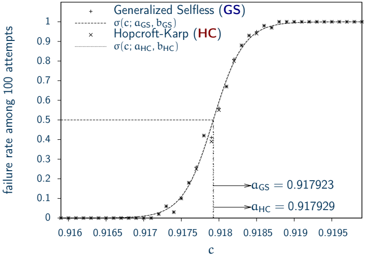

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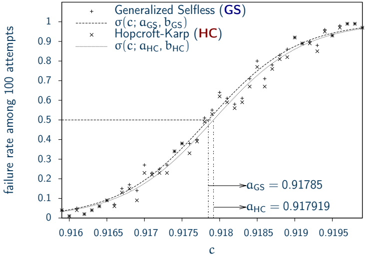

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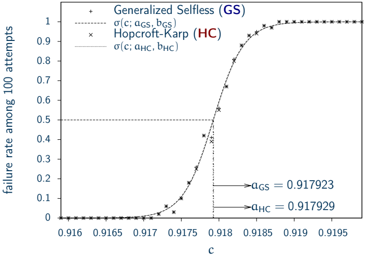

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$m \backslash c$	0.916	0.917	0.918	0.919

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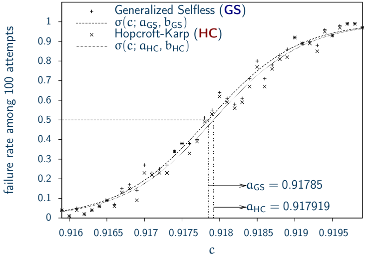

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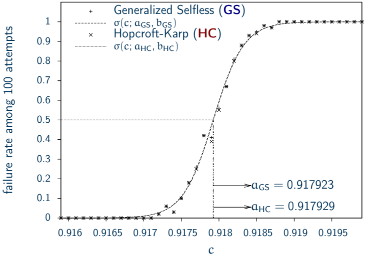

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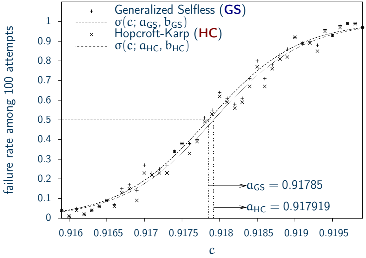

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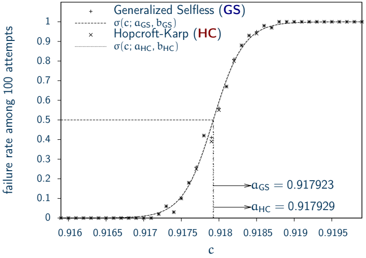

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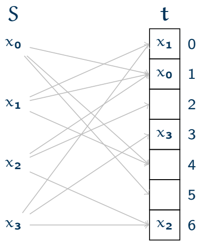
10^6	**1.74** **9.34**	**1.75** **10.99**	**1.76** **16.36**	**1.76** **16.85**

Summary

Performance:

- space utilization
  - membership: $1/c \cdot n \cdot \log|U|$



d	2	3	4	5	...	$O(\log(n))$
$1/c$	2.01	1.09	1.03	1.01	...	1

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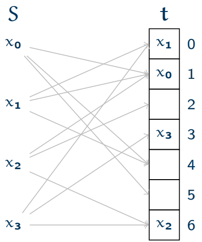

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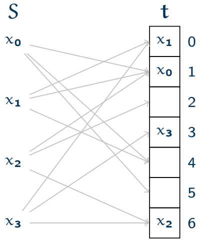

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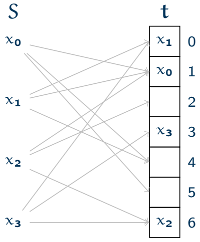

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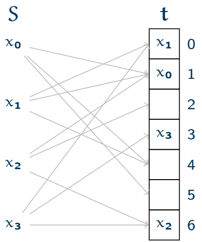

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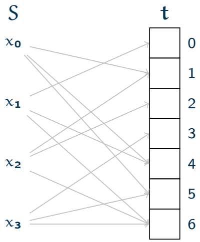
Open: Proof that if $\mathcal{B}$ admits a matching, then whp the Generalized Selfless Algorithm finds a matching.

Next . . .

Retrieval Data Structure

Bloomier Filter [Chazelle, Kilian, Rubinfeld, Tal, 2004],
Basic Retrieval Data Structure [Dietzfelbinger and Pagh, 2008]**:**



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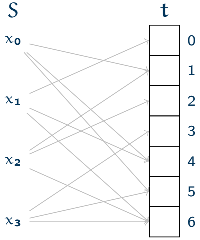
$(V, \oplus) = (\mathbb{Z}_6, +)$



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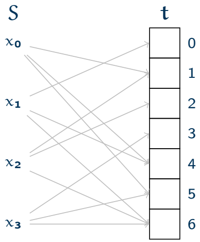

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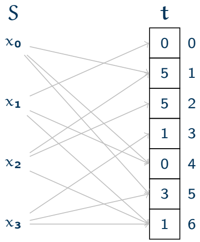

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- $\mathtt{lookup}(\mathfrak{D}, x) \coloneqq \bigoplus_{a\in A_x} t_a$

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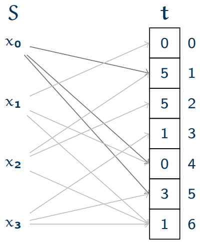
$(5+0+3) \bmod 6=2$



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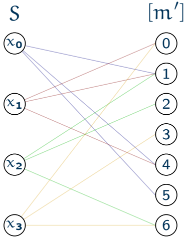

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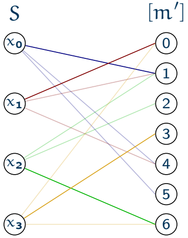

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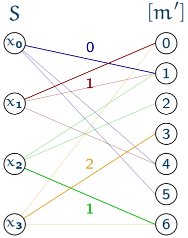
- build vector v of indices $\iota(x)$, where $\{x, h'_{\iota(x)}\}$ is matching edge



$v = (0, 1, 1, 2)$

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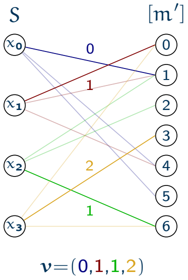
- build retrieval data structure for $\boldsymbol{v}$



Requirements (1)

Construction possible:

$\Leftarrow$ M has full row rank n, i.e. $n \times n$ submatrix with non-zero determinant in $\mathbb{F}_2$

Connections

Theorem: [Dietzfelbinger et al., 2010] based on [Dubois and Mandler, 2002]

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Question: How can we reach linear running time?

Requirements (2)

Construction possible:

$\Leftarrow$ M has full row rank n

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline x_0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ x_1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ x_2 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ x_3 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

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$$
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\end{bmatrix}
\end{array}
$$

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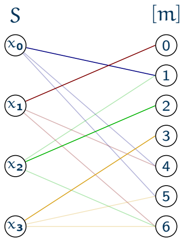

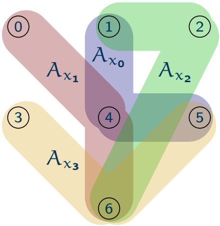

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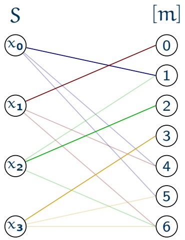

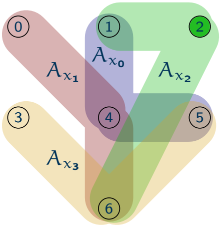

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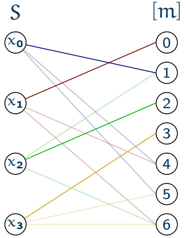

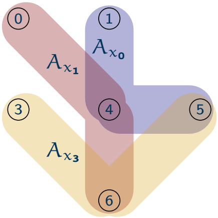

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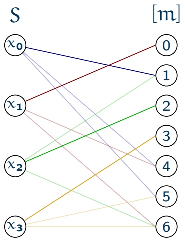

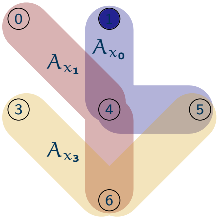

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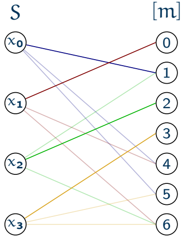
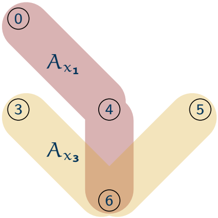


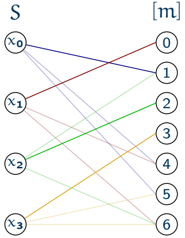


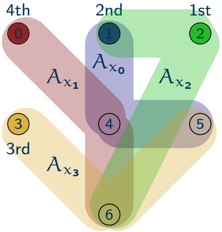
Requirements (2)

Construction possible:

- $\Leftarrow$ $\mathbf{M}$ has full row rank n
- $\Leftrightarrow$ elementary operations transform $\mathbf{M}$ in row echelon form
- $\Leftarrow$ *only* row and column permutations transform $\mathbf{M}$ in row echelon form
- $\Leftrightarrow$ 2-core of $\mathcal{H}$ is empty

$$\begin{array}{c|c} & \begin{array}{ccccccc} 2 & 1 & 3 & 0 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} x_2 \\ x_0 \\ x_3 \\ x_1 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$





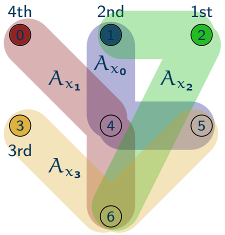
Peeling and Back-substitution

Greedy Algorithm:

Peeling and Back-substitution

Greedy Algorithm:

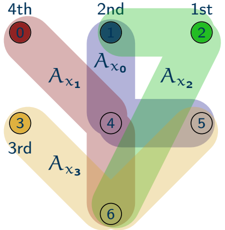
- determine row and column permutations



Peeling and Back-substitution

Greedy Algorithm:

- determine row and column permutations
- apply back-substitution



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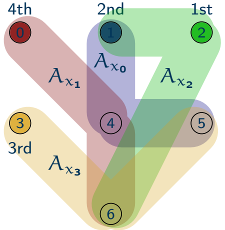
\end{array}$$

Peeling and Back-substitution

Greedy Algorithm:

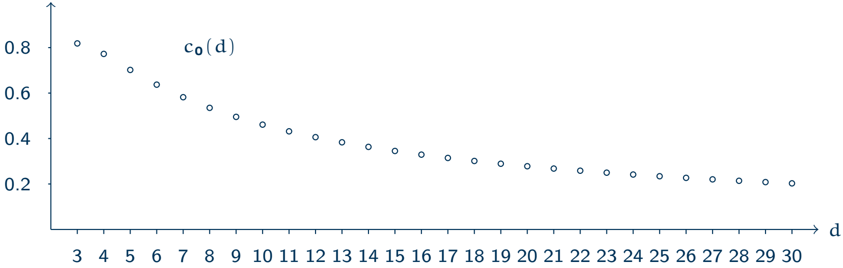
- determine row and column permutations
- apply back-substitution
- running time $O(n)$



$$\begin{array}{c c} & \begin{matrix} 2 & 1 & 3 & 0 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} x_2 \\ x_0 \\ x_3 \\ x_1 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Thresholds

Appearance of 2-core [Molloy, 2004],[Cooper, 2004],[Kim, 2006],...



New: Optimality

Question: Can we beat $c_0(3) \approx 0.8185$ using different edge sizes?

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For two edge sizes d_0 and d_1 the maximum threshold

$$c_0(d_0, d_1) \coloneqq \max_{\alpha} c_0((d_0, d_1), (\alpha, 1 - \alpha))$$

can be calculated efficiently, and for appropriate d_0 and d_1, this value is larger than $c_0(3)$.

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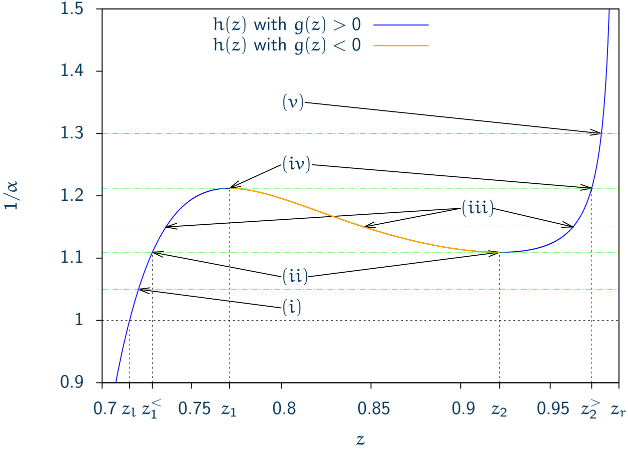
Proof idea:

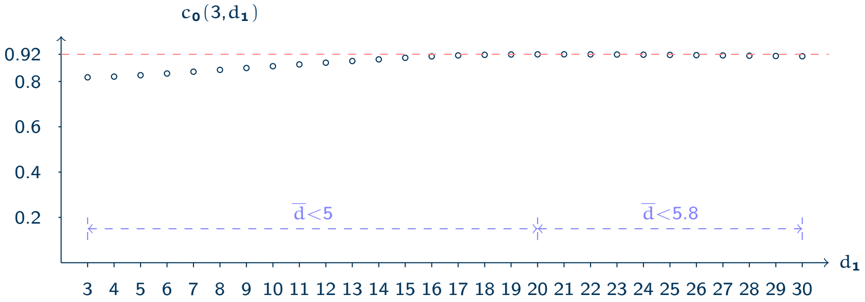
* multivariate calculus, non-convex optimization

Non-convex Optimization



- identify *critical points* in z-direction
- determine $z'(d_0, d_1)$, the maximum point of the *function of critical points*
- find z with smallest distance to z' that is *legal global minimum point*

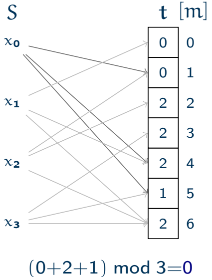
Thresholds for Mixed Degrees



Summary

Performance: e.g. using factor $1.1 > \dfrac{1}{c_0(3,16)}$

- space utilization
  - retrieval DS: $1.1 \cdot n \cdot \log|V|$



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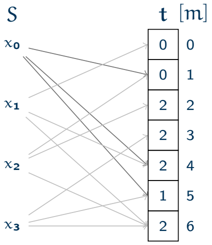

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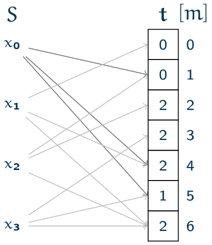

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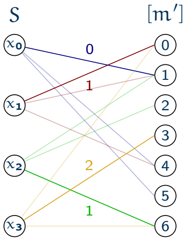
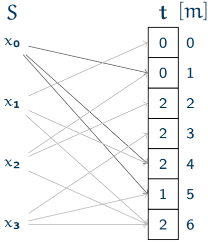


$v=(0,1,1,2)$



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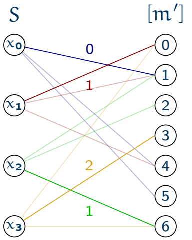

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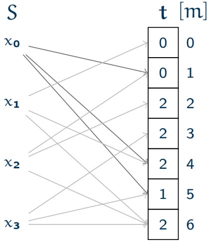

(0+2+1) mod 3=0

Summary

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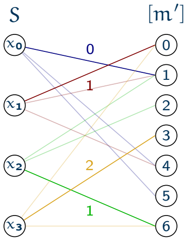
    - $1.1 \cdot n \cdot \lceil \log 3 \rceil$
    - $1.1 \cdot n \cdot 8/5$ (simple compression)
- construction time: $O(n)$ (expected)
- cell probes: average < 6, worst-case 16



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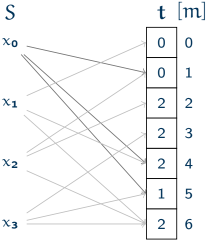

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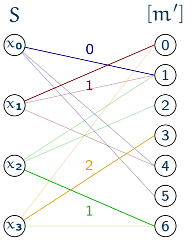
Open: Show that c_0 for mixed edge sizes can be arbitrary close to 1.



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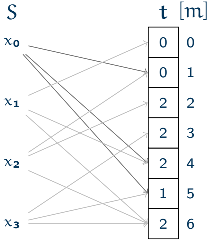

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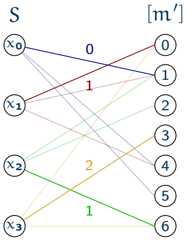
Open: Given $\overline{d}$, determine mix of edge sizes that maximizes c_0.



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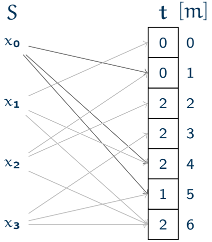

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Thank you!

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