# Towards Optimal Degree Distributions for Left-perfect Matchings in Random Bipartite Graphs

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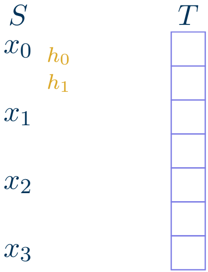
# Introduction

# Motivation

multiple choice hash table

- set $S$ with $n$ keys, table $T$ with $m$ cells
- hash functions $h_0, h_1, \ldots, h_{d-1}$



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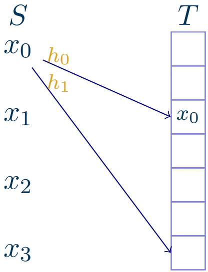

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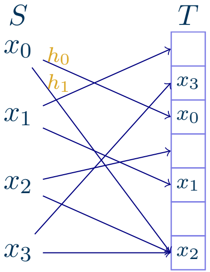

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- expected lookup time $(d+1)/2$
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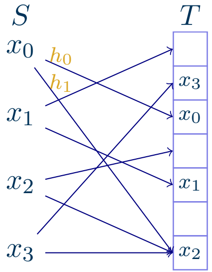

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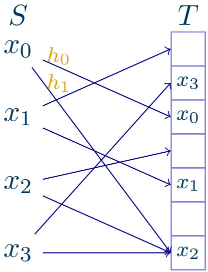


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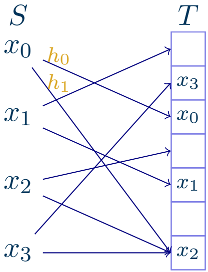


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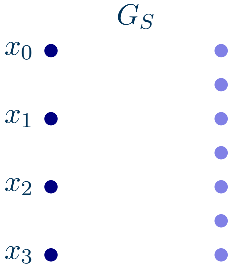
**Question**

Can we do better if the keys are not restricted to use the same number of possible table cells?

# Graph Model

bipartite graph $G_S$

- $n$ left nodes, labeled with elements from $S$, $m$ right nodes
- for each $x$ from $S$ the degree is a random variable $D_x$



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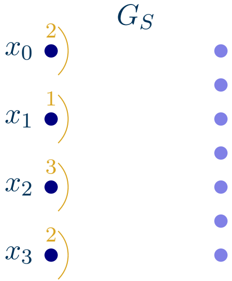

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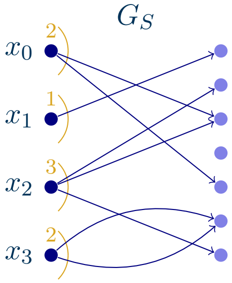

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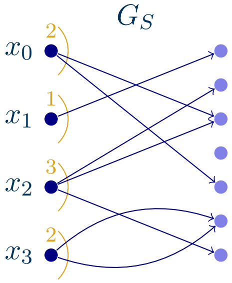

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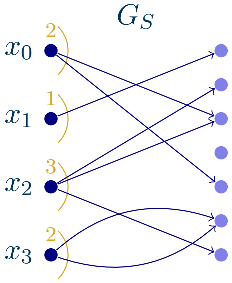
- each $D_x$ follows its own probability mass function $\rho_x$



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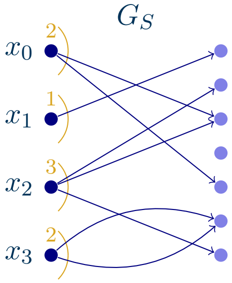
- $\Delta_x$ is the expectation of $D_x$
- $\bar{\Delta}$ is the average expected value



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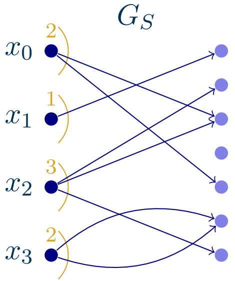


**Question**

Given parameters $(n, m, \bar{\Delta})$, which sequence of probability mass functions $(\rho_x)_{x \in S}$ maximizes the probability that $G_S$ has a left-perfect matching?

# Main Result

**Theorem**

Let $n \leq m$, as well as $n, \bar{\Delta} \geq 2$, and let $(\rho_x)_{x \in S}$ be an optimal sequence for parameters $(n, m, \bar{\Delta})$. Then the following holds for all $x \in S$.

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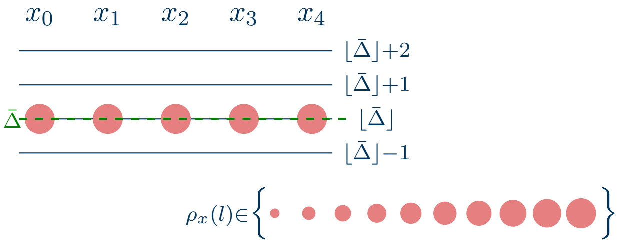
$\bar{\Delta}$ is an integer



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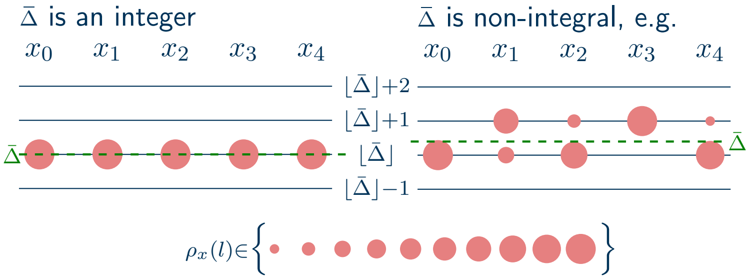
- If $\bar{\Delta}$ is an integer, then $\rho_x(\bar{\Delta}) = 1$.
- If $\bar{\Delta}$ is non-integral, then $\rho_x(\lfloor\bar{\Delta}\rfloor) \in [0, 1]$ and $\rho_x(\lceil\bar{\Delta}\rceil) = 1 - \rho_x(\lfloor\bar{\Delta}\rfloor)$.



# Related Work

generalized Cuckoo Hashing (similar graphs, pairwise distinct neighbors)

- thresholds for regular case — 1 fixed degree
  - ⋆ [Frieze and Melsted, 2009]
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erasure correcting codes (different graphs)

- uneven distribution increases probability for existence of a matching that can be calculated in linear time (peeling)
  - Tornado Codes [Luby, Mitzenmacher, Shokrollahi, Spielman, 2001]
  - LT Codes [Luby, 2002]
  - Online Codes [Maymounkov, 2002]
  - Raptor Codes [Shokrollahi, 2006]

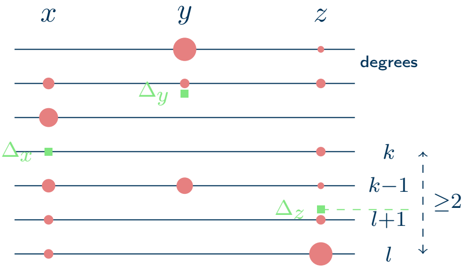
  ...

# Proof of Main Theorem

# Degrees Must be Next to the Mean

**Lemma**

Let $(\rho_x)_{x\in S}$ be given. Let $z \in S$ be arbitrary but fixed. If in $\rho_z$ two degrees with distance at least $2$ have nonzero probability then $(\rho_x)_{x\in S}$ is not optimal.



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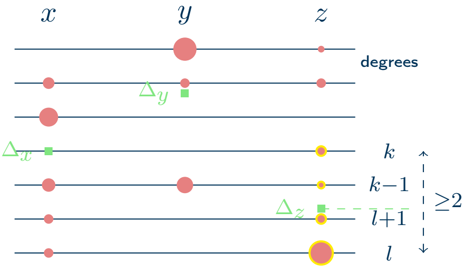


For some $\varepsilon > 0$

- increase the probability of degrees $l+1$ and $k-1$ by $\varepsilon$
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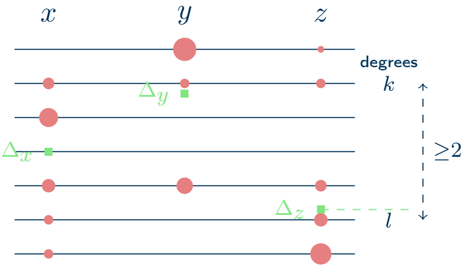


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This decreases the failure probability but does not change $\Delta_z$ and $\bar{\Delta}$.

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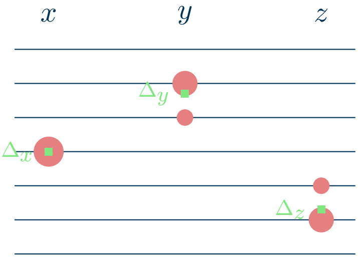


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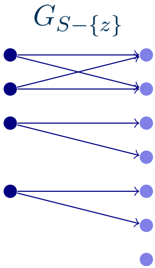
Fix graph without node $z$. $G_{S-\{z\}}$ may admit many matchings.
Subdivide right node set into



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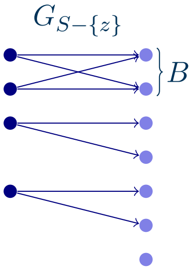

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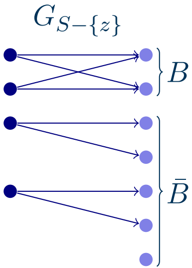

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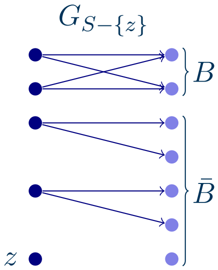


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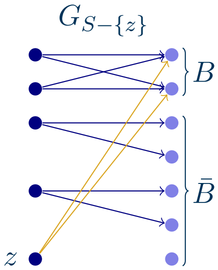


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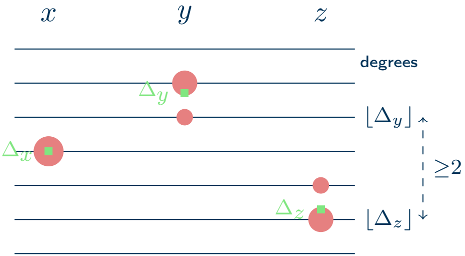
$$N_z := \{u \mid (z,u) \text{ is edge in } G_S\} \subseteq B$$

# Average Degrees of Different Nodes are Close

**Lemma**

Let $(\rho_x)_{x \in S}$ be given, where for each $x \in S$ only degrees from $\{\lfloor \Delta_x \rfloor, \lceil \Delta_x \rceil\}$ have nonzero probability. Let $y, z \in S$ be arbitrary but fixed. If $\lfloor \Delta_y \rfloor$ and $\lfloor \Delta_z \rfloor$ have distance at least $2$, or $\lceil \Delta_y \rceil$ and $\lceil \Delta_z \rceil$ have distance at least $2$, then $(\rho_x)_{x \in S}$ is not optimal.



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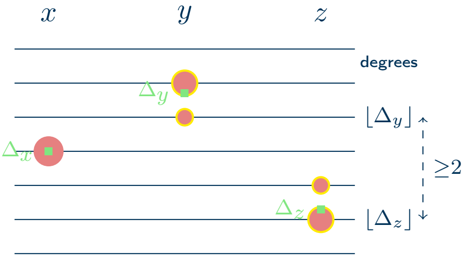


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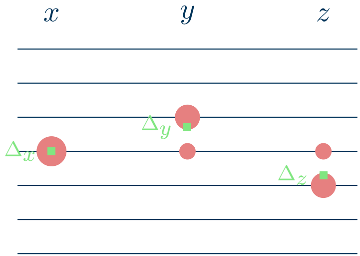


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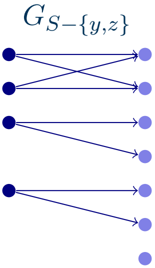

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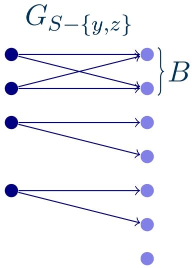

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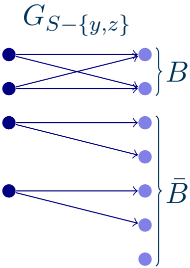

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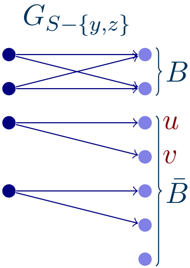
Define relation $\sim$ for nodes from $\bar{B}$:

**$u \sim v$ iff**

there is no matching of $G_{S-\{y,z\}}$ that leaves $u$ and $v$ simultaneously unmatched



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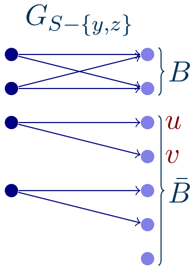


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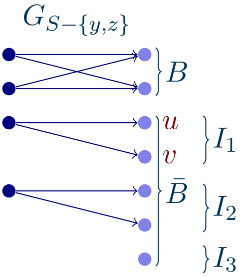


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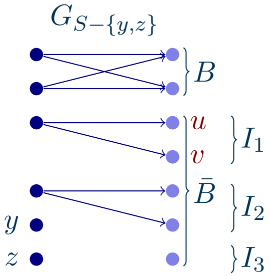


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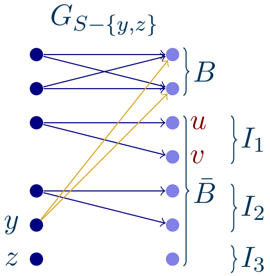
- $B$: blocked nodes, nodes that are matched in all matchings
- $\bar{B}$: non-blocked nodes

Define relation $\sim$ for nodes from $\bar{B}$:

**$u \sim v$ iff**

there is no matching of $G_{S-\{y,z\}}$ that leaves $u$ and $v$ simultaneously unmatched




Prove that $\sim$ is an equivalence relation.

Define sets $I_j$ as equivalence classes of $\sim$.

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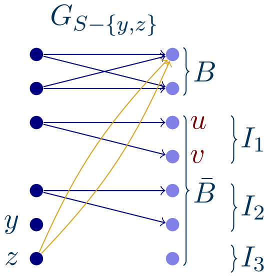


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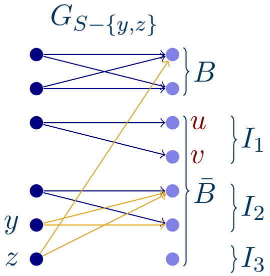


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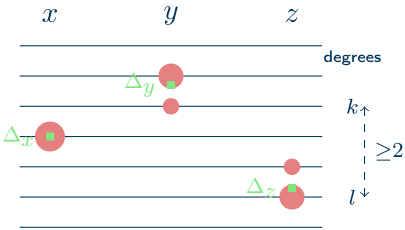
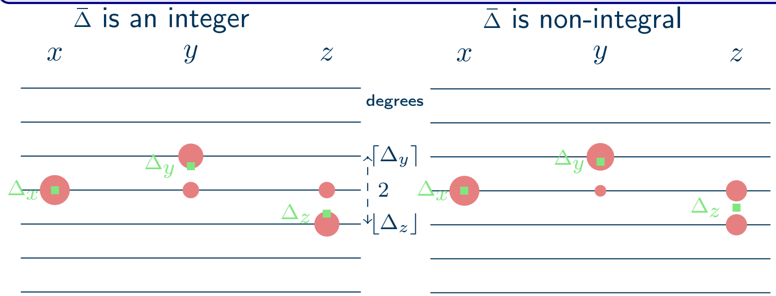
for each vector $(b, r, i_1, \dots, i_r)$.



# Optimal Distributions Use At Most Two Neighboring Degrees

**Lemma**

Let $(\rho_x)_{x\in S}$ be given as in the last lemma. Let $y, z \in S$ be arbitrary but fixed and assume that $\Delta_y$ and $\Delta_z$ are non-integral. If $\lceil \Delta_y \rceil$ and $\lfloor \Delta_z \rfloor$ have distance $2$ then $(\rho_x)_{x\in S}$ is not optimal.



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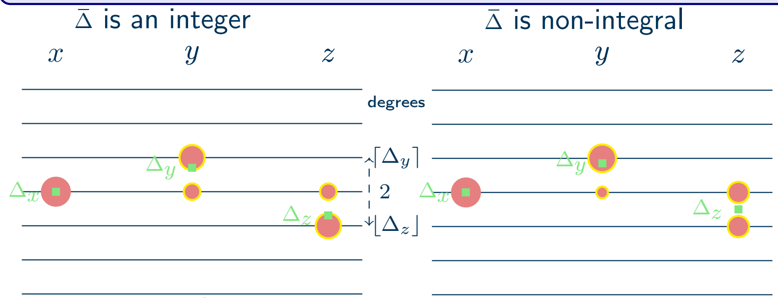


For some perturbation $\varepsilon \neq 0$

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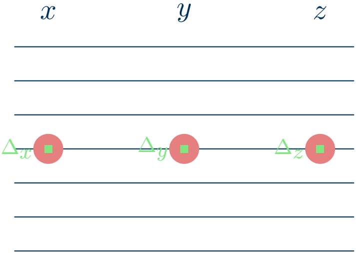


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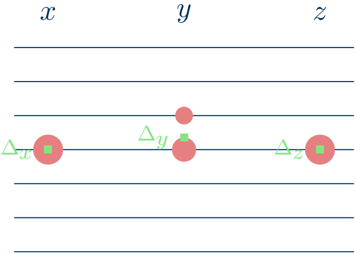


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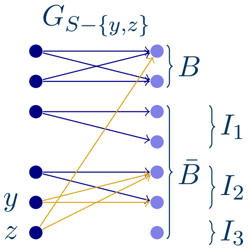
This decreases the failure probability but does not change $\bar{\Delta}$.

# Proof Idea

Same approach as before — $b$ blocked nodes, $r$ equivalence classes of size $i_j$



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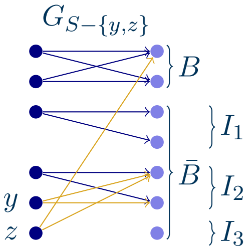


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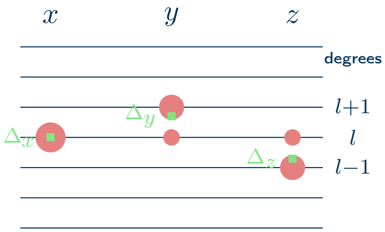

# Conjecture

# Non-integral Case

optimal distribution could be

sufficient condition

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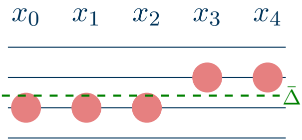


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$$\mathrm{fail}(l,l) - \mathrm{fail}(l+1,l)$$
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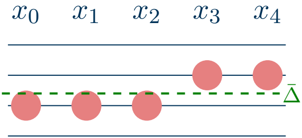


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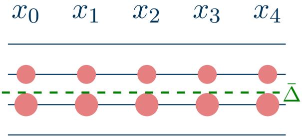


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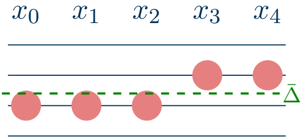


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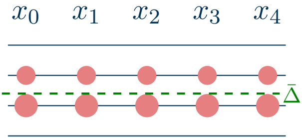


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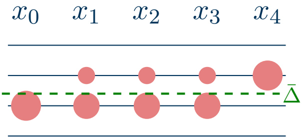


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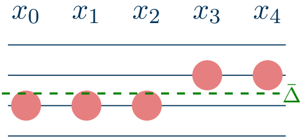


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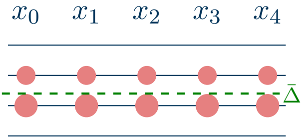


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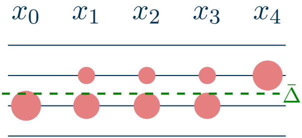


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Not easy to decide. Consider special case $n = \Theta(m)$.

# Existence of Thresholds

**Proposition**

Let $n = c \cdot m$, for constant $c > 0$, and let $(\rho_x)_{x \in S}$ be a near optimal sequence with average expected degree $\bar{\Delta} > 2$. Then for sufficiently large $n$ there is a threshold $c^*(\bar{\Delta})$ such that the random graph $G_S$ has the following property.

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This follows directly by the proofs for the thresholds of $k$-ary cuckoo hashing (with integral and non-integral $k = \bar{\Delta}$) using standard techniques on concentration bounds for nodes of certain degrees.

# Experimental Failure Rate Around the Threshold

$(n, m, \bar{\Delta})$ is fixed, only two possible degrees $\lfloor \bar{\Delta} \rfloor$ and $\lceil \bar{\Delta} \rceil$

$\alpha$ is fraction of left nodes with degree $\lfloor \bar{\Delta} \rfloor$

$f(\alpha)$ is rate of random graphs that have no matching — failure rate

$$c^*(\bar{\Delta}) = 0.957$$

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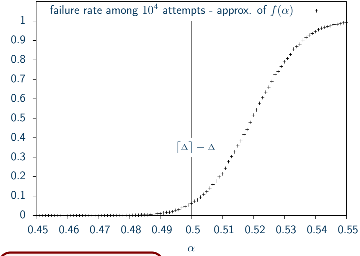


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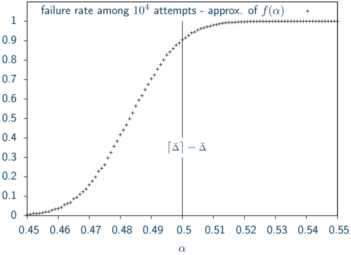


**Conjecture**

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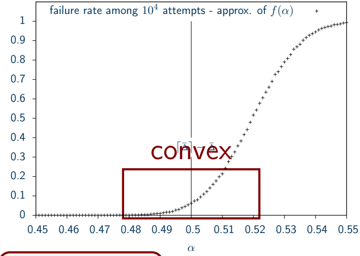


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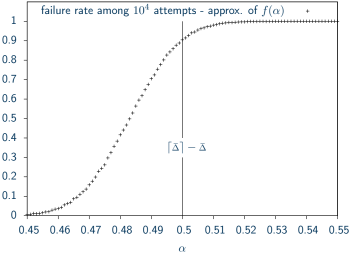


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If $c$ is to the left of $c^*(\bar{\Delta})$ then it is optimal to fix the degrees.

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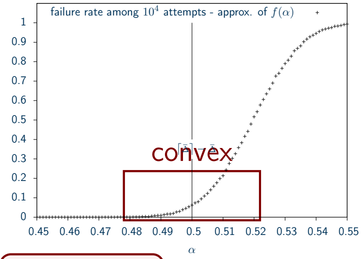


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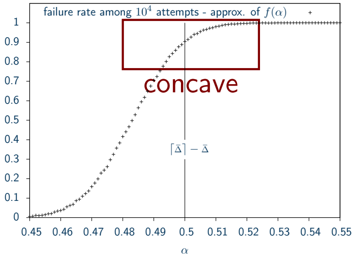


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If $c$ is to the left of $c^*(\bar{\Delta})$ then it is optimal to fix the degrees.
If $c$ is to the right of $c^*(\bar{\Delta})$ then it is optimal to use the Binomial distribution.

# Conclusion

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## Open problem

Prove the conjecture.

Thank you!

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